Dr. Franzke

EE 364 - Fall 2026

Week 6

Supplemental Notes

Moments

(a) Population statistics

population mean: $\mu_x = E_x[X] = \begin{cases} \sum_x x \cdot f(x) & \text{discrete} \\ \int_{-\infty}^{\infty} x \cdot f(x)\,dx & \text{continuous} \end{cases}$ (X: r.v.)

if sum/integral exists (i.e. finite)

$E_x[X]$ "averages out the randomness"

(b) vs. Sample statistics

sample mean: $\bar{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k$ ($\bar{X}_n$: r.v.; X_k: r.v.)

(realizations: $\bar{x}_n = \frac{1}{n}\sum_{k=1}^{n} x_k$)

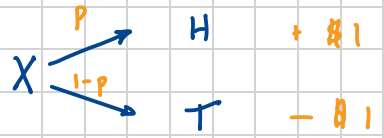
Defn: The <u>population mean</u> (expectation)

$$\mu_x = E_x[X] = \begin{cases} \sum_x x \cdot f_x(x) & X \text{ discrete} \\ \int_{-\infty}^{\infty} x \cdot f_x(x)\,dx & X \text{ continuous} \end{cases}$$

(μ_x: constant; X: r.v.)

if sum/integral exists (is finite)

Ex: Coin flip $\Omega = \{H, T\}$, $\mathcal{A} = 2^{\Omega}$



reminder: $(\Omega, \mathcal{A}, P)$ in background.

$$E_x[X] = P[H] \cdot x_H + P[T] \cdot x_T$$
$$= p(\$1) - q(\$1)$$

$\therefore$ if $p = q \longrightarrow E_x[X] = 0$

Aside: Casino: $E_{You}[X] < 0$

Rational price — price = E[gain]

"break even" on average

vs. min[loss]

Thm: (Law of the unconscious statistician, LOTUS)

X random variable, $Y = g(X)$

$$E_Y[Y] = E_X[g(X)] = \begin{cases} \int_{-\infty}^{+\infty} g(x) \cdot f_x(x)\, dx \\ \sum_x g(x) \cdot p_x(x) \end{cases} \quad \text{if } E[|X|] < \infty$$

Defn: The <u>population variance</u>

$$\sigma_x^2 = V_x[X] = E_x[(X - E_x[X])^2].$$

if finite $(\sigma_x^2 < \infty)$

Note: Cauchy $\sigma_x^2 = \infty$.

Thm: If $X \sim b(n,k,p)$ (Binomial)

① $E[X] = n \cdot p$

② $V[X] = n \cdot p \cdot q$

Pf: ① $E[X] = \sum_{k=0}^{n} k \cdot P[X=k] = \sum_{k=0}^{n} k \cdot b(n,k,p)$

$\overset{X \sim b}{=} \sum_{k=0}^{n} k \binom{n}{k} p^k q^{n-k} = \sum_{k=1}^{n} k \cdot \frac{n!}{k!(n-k)!} p^k q^{n-k}$

$= \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k q^{n-k}$ put $j = k-1$ $\therefore k = j+1$

$= \sum_{j+1=1}^{j+1=n} \frac{n(n-1)!}{j!(n-1-j)!} p^{j+1} q^{(n-1)-j}$

$= \sum_{j=0}^{n-1} \frac{n \cdot (n-1)!}{j!((n-1)-j)!} p \cdot p^j q^{(n-1)-j}$

$= n \cdot p \sum_{j=0}^{n-1} \frac{(n-1)!}{j!((n-1)-j)!} p^j q^{(n-1)-j}$ (binomial theorem $= 1$)

$= n \cdot p.$

② $E[X^2] = \sum_{k=0}^{n} k^2 \cdot \frac{n!}{k!(n-k)!} p^k q^{n-k}.$

$= n \cdot p \cdot \sum_{k=1}^{n} k \cdot \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} q^{n-k}$ put $j = k-1$, $k = j+1$

$= n \cdot p \cdot \sum_{j=0}^{n-1} (j+1) \frac{(n-1)!}{j!((n-1)-j)!} p^j q^{(n-1)-j}$

$= np \cdot \left[\sum_{j=0}^{n-1} j \cdot \frac{(n-1)!}{j!((n-1)-j)!} p^j q^{(n-1)-j} \right.$ $= (n-1)p$ by ①

$\left. + \sum_{j=0}^{n-1} \frac{(n-1)!}{j!((n-1)-j)!} p^j q^{(n-1)-j} \right]$ $= 1$ by B.T.

$= np \cdot [(n-1)p + 1] = np(np + (1-p))$

$= np(np+q) = (np)^2 + npq$

$\therefore V[X] = E[X^2] - E^2[X]$

$= \cancel{(np)^2} + npq - \cancel{(np)^2} = npq$

QED

Proposition 1: If $Y = aX + b$ $(a \neq 0)$ and $X \sim f_X$

then:

① $E_Y[Y] = a \cdot E_X[X] + b = a \cdot \mu_X + b$

② $V_Y[Y] = a^2 \cdot V_X[X] = a^2 \cdot \sigma_X^2$

Prf: prove for continuous, discrete replace integral / sum

① $E_Y[Y] = E_X[aX + b] = \int_{-\infty}^{+\infty} (ax + b) f_X(x)\,dx$ (if: continuous)

$$= \int_{-\infty}^{+\infty} \left(a x f_X(x) + b f_X(x)\right) dx$$

$$= a \underbrace{\int_{-\infty}^{+\infty} x \cdot f_X(x)\,dx}_{= E_X[X]} + b \cdot \underbrace{\int_{-\infty}^{\infty} f_X(x)\,dx}_{= 1}$$

$$= a \cdot E_X[X] + b$$

② $V_Y[Y] = V_X[aX + b] = E_X\left[\left(aX + b - E[aX + b]\right)^2\right]$

$$= E_X\left[\left(aX + \cancel{b} - aE_X[X] - \cancel{b}\right)^2\right]$$

$$= E_X\left[\left(a(X - E_X[X])\right)^2\right]$$

$$= E_X\left[a^2 (X - E_X[X])^2\right]$$

$$= a^2 \cdot \underbrace{E_X\left[(X - E_X[X])^2\right]}_{V_X[X]}$$

$$= a^2 \cdot V_X[X]$$

Proposition 2: $V_X[X] = E_X[X^2] - E_X^2[X]$.

Prf: $V_X[X] = E\left[(X - E_X[X])^2\right] = E_X\left[X^2 - 2 \cdot X \cdot \underbrace{E_X[X]}_{\text{constant}} + \underbrace{(E_X[X])^2}_{\text{constant.}}\right]$.

$$= E_X[X^2] - 2 \cdot \underbrace{E_X[X] \cdot E_X[X]}_{(E_X[X])^2} + \left(E_X[X]\right)^2$$

$$= E_X[X^2] - \left(E_X[X]\right)^2$$

Proposition 3: ("Standardization") If $Z = \frac{X-\mu}{\sigma}$ $(\sigma_X^2 < \infty)$

then: ① $E[Z] = 0$

② $V[Z] = 1$

Prf: ① $E_z[Z] = E_x\left[\frac{X - E[X]}{\sqrt{V[X]}}\right] = E_x\left[\frac{X-\mu}{\sigma}\right]$

$= E_x\left[\frac{1}{\sigma} \cdot X - \frac{\mu}{\sigma}\right]$

$= \frac{1}{\sigma} \cdot \underbrace{E_x[X]}_{\mu} - \frac{\mu}{\sigma}$

$= \frac{\mu}{\sigma} - \frac{\mu}{\sigma}$

$= 0.$

② $V_z[Z] = V_x\left[\frac{1}{\sigma} X + \frac{\mu}{\sigma}\right] = \frac{1}{\sigma^2} \cdot \underbrace{V_x[X]}_{\sigma^2} = \frac{\sigma^2}{\sigma^2} = 1.$

Thm: If $X \sim G(p)$ $\quad \mu_X = 1/p$, $\sigma_X^2 = q/p^2$ — Geometric, $0<p<1$

$\therefore$ If $X \sim NB$: $\quad \mu_X = k/p$, $\sigma_X^2 = kq/p^2$ — Negative binomial

since $X = X_1 + \cdots + X_k$ and independent $X_j \sim G(p)$.

Prf: ① $\mu_X = E[X] \overset{G(p)}{=} \sum_{n=1}^{\infty} n \cdot p \cdot q^{n-1} = p \sum_{n=1}^{\infty} n \cdot q^{n-1}$

$= p \cdot \sum_{n=1}^{\infty} \frac{d}{dq} q^n \overset{\text{R.O.C.}}{=} p \cdot \frac{d}{dq}\left(\sum_{n=1}^{\infty} q^n\right)$

since R.O.C. $p<1$

$\therefore$ uniform convergence ($\therefore$ commute)

$= p \cdot \frac{d}{dq}\left(\frac{q}{1-q}\right) = p \cdot \frac{(1-q)-q}{(1-q)^2}$

$= \frac{p}{(1-q)^2} \overset{p=1-q}{=} \frac{p}{p^2} = \frac{1}{p}$ since $p>0$

② $E[X^2] = \sum_{n=1}^{\infty} n^2 \cdot p \cdot q^{n-1} = p \cdot \sum_{n=1}^{\infty} n^2 q^{n-1}$

$= p \cdot \sum_{n=1}^{\infty} \frac{d}{dq}(n \cdot q^n) \overset{\text{R.O.C. u.c.}}{=} p \cdot \frac{d}{dq}\left(\sum_{n=1}^{\infty} n \cdot q^n\right)$

$= p \cdot \frac{d}{dq}\left(q \cdot \sum_{n=1}^{\infty} n \cdot q^{n-1}\right)$

$= p \cdot \frac{d}{dq}\left(\frac{1}{p} \sum_{n=1}^{\infty} n \cdot p \cdot q^{n-1}\right)$

$= p \cdot \frac{d}{dq} \cdot \left(\frac{q}{p} \cdot E[X]\right)$

$\overset{①}{=} p \cdot \frac{d}{dq}\left(\frac{q}{p^2}\right) = p \cdot \frac{d}{dq}\left(\frac{q}{(1-q)^2}\right)$

$= p \cdot \frac{(1-q)^2 + 2q(1-q)}{(1-q)^4}$ Quotient rule

$= \frac{p}{(1-q)^2} + \frac{2pq}{(1-q)^3} = \frac{p}{p^2} + \frac{2pq}{p^3}$

$p > 0$

$$= \frac{1}{p} + \frac{2(1-p)}{p^2} = \frac{1}{p} + \frac{2}{p^2} - \frac{2p}{p^2} = \frac{1}{p} - \frac{2}{p^2} - \frac{2}{p}$$

$$= \frac{2}{p^2} - \frac{1}{p}$$

$$\therefore \sigma_X^2 = E[X^2] - E^2[X] = \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2}$$

$$= \frac{1}{p^2} - \frac{1}{p} = \frac{1-p}{p^2}$$

$$= \frac{q}{p^2}$$

QED.

Thm: If $X \sim U[a,b]$ ① $E[X] = \frac{a+b}{2}$ Uniform.

② $V[X] = \frac{(b-a)^2}{12}$

Pf: ① $E_x[X] = \int_{-\infty}^{+\infty} x \, f_x(x)\, dx = \int_a^b x \, \frac{1}{b-a}\, dx$

$f_x(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{else} \end{cases}$

$= \frac{1}{b-a} \int_a^b x \, dx = \frac{1}{b-a} \left(\frac{x^2}{2} \Big|_{x=a}^{x=b} \right)$

$= \frac{1}{b-a} \cdot \frac{(a+b)(b-a)}{2}$

$= \frac{a+b}{2}$

② $E_x[X^2] = \int_{-\infty}^{+\infty} x^2 \cdot f_x(x)\, dx = \int_a^b x^2 \cdot \frac{1}{b-a}\, dx$

$= \frac{1}{b-a} \int_a^b x^2 \, dx = \frac{1}{b-a} \left(\frac{x^3}{3} \Big|_{x=a}^{x=b} \right)$

$= \frac{1}{b-a} \cdot \frac{(b-a)(b^2+ab+a^2)}{3}$

$= \frac{b^2+ab+a^2}{3}$

$\therefore V_x[X] = E_x[X^2] - (E_x[X])^2$

$= \frac{4}{4} \cdot \frac{b^2+ab+a^2}{3} - \left(\frac{a+b}{2} \right)^2 \cdot \frac{3}{3}$

$= \frac{4b^2+4ab+4a^2}{12} - \frac{3a^2+6ab+3b^2}{12}$

$= \frac{b^2-2ab+a^2}{12}$

$= \frac{(b-a)^2}{12}$

Thm: If $X \sim C[m, d]$ ① $E[X] =$ (does not exist) Cauchy

② $V[X] =$ (∞)

Prf: ① $E_x[X] = \int_{-\infty}^{\infty} x \cdot f_x(x)\, dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{\pi d\left(1+\left(\frac{x-m}{d}\right)^2\right)}\, dx.$

Let $u = \frac{x-m}{d}$ $\quad du = \frac{dx}{d}$

$$= \int_{-\infty}^{\infty} (ud + m)\, \frac{1}{\pi(1+u^2)}\, du$$

$$= \frac{d}{\pi} \cdot \int_{-\infty}^{+\infty} \frac{u}{1+u^2}\, du + \frac{m}{\pi} \underbrace{\int_{-\infty}^{+\infty} \frac{1}{1+u^2}\, du}_{\text{pdf}_t = 1.}$$

$$= \frac{d}{\pi} \underbrace{\int_{-\infty}^{+\infty} \frac{u}{1+u^2}\, du} + \frac{m}{\pi}$$

$$\int_{-\infty}^{\infty} \left|\frac{u}{1+u^2}\right| du \le \int_{-\infty}^{\infty} \left|\frac{u}{u^2}\right| du$$

$$= \int_{-\infty}^{+\infty} \left|\frac{1}{u}\right| du = \infty$$

② $E_x[X^2] = \int_{-\infty}^{\infty} x^2 \cdot f_x(x)\, dx = \int_{-\infty}^{+\infty} x^2 \cdot \frac{1}{1+x^2}\, dx = \int_{-\infty}^{+\infty} \frac{x^2}{1+x^2}\, dx$

$$\le \int_{-\infty}^{+\infty} \frac{x^2}{x^2}\, dx$$

$$= \int_{-\infty}^{+\infty} 1\, dx = \infty.$$

Note: Variance $\sigma_x^2 = E_x[(X-\mu_x)^2]$ type of dispersion.

("average", "distance")

Variance ≠ Dispersion, e.g., could use $|\cdot|$ (absolute value) instead of $(\cdot)^2$

↳ but math more difficult.

Defn: The k^{th} higher order moment

$$E_x[X^k] = \int_{-\infty}^{\infty} x^k \cdot f_x(x)\,dx \qquad (\text{if} < \infty)$$

In general: k^{th} moment exists $\not\Longleftarrow$ $(k-1)^{th}$ moment exists

$$\therefore \quad \sigma_x^2 < \infty \implies |\mu_x| < \infty$$

Thm: If $k \le m$ and $k \in \mathbb{Z}^+$ and $m \in \mathbb{Z}^+$ then

$E[X^m]$ exists $\longrightarrow$ $E[X^k]$ exists.

Pf: (continuous case, replace integrals with sums for discrete)

$$\int_{-\infty}^{\infty} |x|^k f_x(x)\,dx = \int_{|x|\le 1} |x|^k \cdot f_x(x)\,dx + \int_{|x|>1} |x|^k \cdot f_x(x)\,dx$$

$$\le \int_{|x|\le 1} |x|^k \cdot f_x(x)\,dx + \int_{|x|>1} |x|^m \cdot f_x(x)\,dx$$

since $k \le m$ and $|x| > 1$

$$\le \int_{|x|\le 1} f_x(x)\,dx + \int_{|x|>1} |x|^m f_x(x)\,dx.$$

since $|x| \le 1 \longrightarrow |x|^k$

$$\le \underbrace{\int_{-\infty}^{\infty} f_x(x)\,dx}_{=1 \text{ (pdf)}} + \int_{|x|>1} |x|^m \cdot f_x(x)\,dx$$

since $\{|x| \le 1\} \subset \mathbb{R}$

$$\leq 1 + \int_{-\infty}^{\infty} |x|^m \cdot f_x(x)\, dx.$$

since $\{|x|>1\} \subset \mathbb{R}$

$$= 1 + E_x[|X|^m].$$

$$< \infty \qquad \text{since } E[X^m] \text{ exists by hypo.}$$

$\therefore$ $E[X^k]$ exists.

QED.

Normal moments

★ **Thm:** If $Z \sim N(0,1)$ (standard normal)

then $$E[Z^k] = \begin{cases} 0 & \text{if } k \text{ odd} \\ (k-1)(k-3)\cdots 5\cdot 3\cdot 1 & \text{if } k \text{ even} \end{cases}$$

In general: If $X \sim N(\mu, \sigma^2)$ then

$$E[(X-\mu_X)^k] = \begin{cases} 0 & \text{if } k \text{ odd} \\ \sigma^k (k-1)(k-3)\cdots 5\cdot 3\cdot 1 & \text{if } k \text{ even} \end{cases}$$

k^{th} "central moment"

Pf: Case 1, k even

$$E[Z^k] = \frac{1}{\sqrt{2\pi}} \int_{z=-\infty}^{z=\infty} z^k e^{-z^2/2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{k-1} \left(z \cdot e^{-z^2/2}\, dz \right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u \cdot dv$$

if $u = z^{k-1}$, $du = (k-1)z^{k-2}dz$; $dv = ze^{-z^2/2}dz$, $v = e^{-z^2/2}$

integration by parts

$$= \frac{1}{\sqrt{2\pi}} \left(uv \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v \cdot du \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\underbrace{\lim_{z\to\infty} z^{k-1} \cdot e^{-z^2/2} \Big|_{-z}^{z}}_{=0,\ \text{L'Hospital's}} + (k-1) \cdot \int_{-\infty}^{\infty} z^{k-2} e^{-z^2/2}\, dz \right.$$

$$= \frac{1}{\sqrt{2\pi}} (k-1) \cdot \int_{-\infty}^{\infty} z^{k-2} e^{-z^2/2} \, dz$$

$$= (k-1) \cdot E[X^{k-2}]$$

$$\overset{k\text{-even}}{=} (k-1)(k-3)(k-5) \cdots 5 \cdot 3 \cdot 1$$

Case 2, k odd

$e^{-z^2/2}$ even function

$\therefore z^k e^{-z^2/2}$ odd function if k odd

$\therefore \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^k e^{-z^2/2} \, dz = 0.$

Ex: exponential moments $X \sim \exp(\theta)$

Thm. $\boxed{E[X^k] = k!\,\theta^k \text{ if } X \sim \exp(\theta)}$

Pf:

$$E[X^k] = \int_{-\infty}^{\infty} x^k \cdot f_x(x)\,dx.$$

$$= \int_0^{\infty} x^k \cdot \left(\frac{1}{\theta} \cdot e^{-x/\theta}\right) dx \qquad \frac{\theta^k}{\theta^k}$$

since $X \sim \exp(\theta)$

$$= \theta^{k-1} \int_0^{\infty} \left(\frac{x}{\theta}\right)^k e^{-x/\theta}\,d\theta$$

$\therefore$ Let $u = \frac{x}{\theta}$ $\quad du = \frac{dx}{\theta}$.

$x = 0 \rightarrow u = 0$
$x = \infty \rightarrow u = \infty$

$\therefore dx = \theta \cdot du$

$$= \theta^{k-1} \cdot \int_0^{\infty} u^k e^{-u}\,\theta \cdot du$$

$$= \theta^k \cdot \underbrace{\int_0^{\infty} u^{(k+1)-1} e^{-u}\,du}_{\Gamma(k+1)}$$

$$= \theta^k \cdot \Gamma(k+1)$$

$$= \theta^k \cdot k! \qquad \text{since } \Gamma(\alpha+1) = \alpha \cdot \Gamma(\alpha).$$

Corr: $k! = E[X^k]$ if $X \sim \exp(1)$.

Ex: If $X \sim \exp(\theta)$, then

$$\mu_X = E[X^1] = \theta$$

$$\therefore \quad \sigma_X^2 = E[X^2] - E^2[X] = 2\theta^2 - \theta^2 = \theta^2$$

More generally:

Thm: $E[X^k] = \theta^k \dfrac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$ $\begin{matrix} x>0 \\ \alpha>0 \\ \theta>0 \end{matrix}$ if $X \sim \gamma(\alpha, \theta)$

Pf:

$$E[X^k] = \int_{-\infty}^{\infty} x^k \cdot f_x(x)\, dx$$

$$= \frac{1}{\Gamma(\alpha)\theta^\alpha} \cdot \int_0^\infty x^k \cdot x^{\alpha-1} e^{-x/\theta}\, dx$$

$$= \frac{1}{\Gamma(\alpha)\theta^\alpha} \int_0^\infty x^{(\alpha+k)-1} e^{-x/\theta}\, dx.$$

Let $x = u \cdot \theta$ $\quad dx = \theta \cdot du.$

$$= \frac{1}{\Gamma(\alpha)\cdot\theta^\alpha} \int_0^\infty (u\cdot\theta)^{(\alpha+k)-1} e^{-u}\, \theta\, du$$

$$= \frac{\theta^{(\alpha+k)}}{\Gamma(\alpha)\cdot\theta^\alpha} \cdot \int_0^\infty \underbrace{u^{(\alpha+k)-1} e^{-u}}_{\Gamma(\alpha+k)}\, du.$$

$$= \theta^k \cdot \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$$

QED.

Checking $\alpha = 1$: $\Gamma(\alpha+k) = \Gamma(k+1) = k\cdot\Gamma(k) = k!$

$$\therefore E[X^k] = \theta^k\, k! \quad \text{if } X \sim \text{Exp}(\theta)$$

Corr: ① $\mu_x = \alpha\theta$ if $X \sim \Gamma(\alpha, \theta)$

② $\sigma_x^2 = \alpha\theta^2$

Pf: $\mu_x = E[X^1] \overset{\text{thm}}{=} \theta^1 \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}$

$= \theta \cdot \frac{\alpha \cdot \Gamma(\alpha)}{\Gamma(\alpha)}$

$= \alpha\theta.$

$\sigma_x^2 = E[X^2] - E^2[X] = \theta^2 \cdot \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} - \alpha^2\theta^2.$

$= \theta^2 \left[\frac{(\alpha+1)\Gamma(\alpha+1)}{\Gamma(\alpha)} - \alpha^2 \right].$

$= \theta^2 \cdot \left[\frac{(\alpha+1)\alpha \cdot \Gamma(\alpha)}{\Gamma(\alpha)} - \alpha^2 \right].$

$= \theta^2 \left[\alpha^2 + \alpha - \alpha^2 \right].$

$= \alpha \cdot \theta^2.$ QED

Ex: $X \sim \chi^2(r) \rightarrow \gamma(\alpha, \theta)$, with $\alpha = \frac{r}{2}$, $\theta = 2$

$\therefore E[X^1] = \theta^k \cdot \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = 2^1 \cdot \frac{\frac{r}{2} \cdot \Gamma(\frac{r}{2})}{\Gamma(\frac{r}{2}+1)} = 2 \cdot \frac{r}{2} = r.$

$E[X^2] = \theta^k \cdot \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = 2^2 \cdot \frac{(\frac{r}{2}+1)(\frac{r}{2})\Gamma(\frac{r}{2})}{\Gamma(\frac{r}{2}+2)} = r \cdot (r+2)$

$= r^2 + 2r$

$\therefore V[X] = E[X^2] - (E[X])^2 = r^2 + 2r - (r)^2 = 2r.$

$= \alpha\theta^2 = \frac{r}{2} \cdot 2^2 = 2r$

Defn: **Beta function** $B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx \quad \alpha>0,\ \beta>0$

$= \dfrac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ (see below)

(optional)

Thm: $E[X^k] = \dfrac{B(\alpha+k,\beta)}{B(\alpha,\beta)}$ if $X \sim \text{Beta}(\alpha,\beta)$ and $k>0$

Prf: $E[X^k] = \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} \underbrace{\int_0^1 x^{(\alpha+k)-1}(1-x)^{\beta-1}\,dx}_{B(\alpha+k,\beta)}$

$= \dfrac{B(\alpha+k,\beta)}{B(\alpha,\beta)}$

Corr: $E[X^k] = \dfrac{\alpha+k-1}{\alpha+\beta+k-1}\cdot E[X^{k-1}]$ $(k\geq 1)$.

Prf: $E[X^k] \overset{\text{thm}}{=} \dfrac{B(\alpha+k,\beta)}{B(\alpha,\beta)}$

$= \dfrac{\Gamma(\alpha+k)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta+k)}\cdot\dfrac{1}{B(\alpha,\beta)}$

$= \dfrac{(\alpha+k-1)\cdot\Gamma(\alpha+k-1)\cdot\Gamma(\beta)}{(\alpha+\beta+k-1)\Gamma(\alpha+\beta+k-1)}\cdot\dfrac{1}{B(\alpha,\beta)}$

$= \dfrac{\alpha+k-1}{\alpha+\beta+k-1}\cdot\dfrac{B(\alpha+k-1,\beta)}{B(\alpha,\beta)}$

$= \dfrac{\alpha+k-1}{\alpha+\beta+k-1}\cdot E[X^{k-1}]$

QED.

Corr. If $X \sim \text{Beta}(\alpha, \beta)$ ① $\mu_X = \frac{\alpha}{\alpha+\beta}$.

② $\sigma_X^2 = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$

Pf:

$$\mu_X = E[X^1] = \frac{B(\alpha+1, \beta)}{B(\alpha, \beta)}$$

$$= \frac{\Gamma(\alpha+1)\,\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}$$

$$= \frac{\alpha \cdot \Gamma(\alpha)\,\Gamma(\beta)}{(\alpha+\beta)\cdot\Gamma(\alpha+\beta)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}$$

$$= \frac{\alpha}{\alpha+\beta}.$$

$$E[X^2] = \frac{B(\alpha+2, \beta)}{B(\alpha, \beta)} = \frac{\Gamma(\alpha+2)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta+2)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)}$$

$$= \frac{(\alpha+1)\cdot\Gamma(\alpha+1)}{(\alpha+\beta+1)\cdot\Gamma(\alpha+\beta+1)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}$$

$$= \frac{(\alpha+1)\,\alpha\cdot\Gamma(\alpha)}{(\alpha+\beta+1)(\alpha+\beta)\cdot\Gamma(\alpha+\beta)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}$$

$$= \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)\cdot}$$

$$\therefore \sigma_X^2 = E[X^2] - E^2[X] = \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)} - \frac{\alpha^2}{(\alpha+\beta)^2}$$

$$= \frac{\alpha(\alpha-1)(\alpha+\beta) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta+1)(\alpha+\beta)^2} = \frac{\alpha^2+\alpha^2\beta+\alpha^2+\alpha\beta-\alpha^2-\alpha^2\beta-\alpha^2}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

$$= \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

QED

(optional)

Thm: $B(\alpha,\beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\,du \qquad \alpha>0,\ \beta>0$

$$= \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Prf: $\Gamma(\alpha)\cdot\Gamma(\beta) = \int_{x=0}^{x=\infty} e^{-x}x^{\alpha-1}\,dx \int_{y=0}^{y=\infty} e^{-y}y^{\beta-1}\,dy$

Fubini $= \int_{y=0}^{y=\infty}\left[\int_{x=0}^{x=\infty} e^{-(x+y)}x^{\alpha-1}y^{\beta-1}\,dx\right]dy$

$$= \int_{y=0}^{y=\infty}\int_{x=0}^{x=\infty} f(x(u,v),y(u,v))\,dx\,dy$$

let: $x = x(u,v) = uv \qquad y = y(u,v) = u(1-v)$ double substitution

Given: $0<x<\infty$ and $0<y<\infty$

$\therefore\ 0<x+y = u\cdot v + u(1-v) = \cancel{uv} + u - \cancel{uv} = u$

$\therefore\ u>0$

$\therefore\ x>0 \longrightarrow uv>0$

$\therefore\ v>0$ since $u>0$

$\therefore\ u<\infty$ since $x = u\cdot v<\infty$ and $v>0$

$\therefore\ 0<u<\infty$ ← first limit of integration

$\therefore\ y>0 \longrightarrow u(1-v)>0 \qquad \therefore 1-v>0$ (since $u>0$)

$\therefore\ v<1 \qquad \therefore\ 0<v<1$ ← second limit of integration

$$\therefore \Gamma(\alpha)\cdot\Gamma(\beta) \overset{\text{CVT}}{=} \int_{v=0}^{v=1}\int_{u=0}^{u=\infty} f(u,v)\cdot\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

(change of variable theorem)

$$= \int_{v=0}^{v=1}\int_{u=0}^{u=\infty} f(u,v)\; u\cdot du\,dv.$$

since $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ = absolute determinant of Jacobian matrix $= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix}$ since $x = u\cdot v$, $y = u - uv$

$$= \left| -\cancel{vu} - u + \cancel{vu} \right| = |-u| = u.$$

$$= \int_{v=0}^{v=1}\int_{u=0}^{u=\infty} e^{-u}(uv)^{\alpha-1}(u(1-v))^{\beta-1}\,u\;du\,dv$$

$$= \int_{v=0}^{v=1}\int_{u=0}^{u=\infty} e^{-u}\,u^{\alpha-1}v^{\alpha-1}u^{\beta-1}(1-v)^{\beta-1}u^{1}\,du\,dv$$

$$= \left[\int_{u=0}^{u=\infty} e^{-u}\,u^{\alpha-1+\beta-1+1}\,du\right]\cdot\left[\int_{v=0}^{v=1} v^{\alpha-1}(1-v)^{\beta-1}\,dv\right]$$

$$= \left[\int_{u=0}^{u=\infty} e^{-u}\,u^{(\alpha+\beta)-1}\,du\right]\cdot\left[\int_{v=0}^{v=1} v^{\alpha-1}(1-v)^{\beta-1}\,dv\right]$$

$$= \Gamma(\alpha+\beta)\cdot B(\alpha,\beta)$$

$$\therefore B(\alpha,\beta) = \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

QED

Thm: ① $E[I_A] = P(A)$

② $V[I_A] = P(A) \cdot P(A^c)$

for any $A \in \mathcal{A}$ in probability space $(\Omega, \mathcal{A}, P)$

Prf: ① $E[I_A] = \int_\Omega I_A(\omega)\, dP$

$= \int_A I_A(\omega)\, dP + \int_{A^c} I_{A^c}(\omega)\, dP$ (since $\Omega = A \cup A^c$)

($I_A = 1$ w.p. A; $I_{A^c} = 0$ w.p. A)

$= \int_A 1 \cdot dP + \int_{A^c} 0 \cdot dP = \int_A dP$ ($\int_{A^c} 0 \cdot dP = 0$)

$= P(A)$

② $V[I_A] = \int_\Omega (I_A(\omega) - E[I_A])^2\, dP$

$= \int_\Omega (I_A(\omega) - P(A))^2\, dP$

$= \int_\Omega (I_A^2(\omega) + P^2(A) - 2 \cdot P(A) \cdot I_A(\omega))\, dP$

$= \int_\Omega I_A^2(\omega)\, dP + P^2(A) \int_\Omega dP - 2 \cdot P(A) \cdot \int_\Omega I_A(\omega)\, dP$

$= \int_\Omega I_A(\omega)\, dP + P^2(A) - 2P^2(A)$

(since $I_A^2 = I_A$ for binary sets (not fuzzy))

$= P[A] - P^2[A] = P(A) \cdot (1 - P(A))$

$= P(A) \cdot P(A^c)$

QED